# Optimal Trend Following Trading Rules*

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July 19, 2011

**Abstract**

We develop an optimal trend following trading rule in a bull-bear switching market, where the drift of the stock price switches between two parameters corresponding to an uptrend (bull market) and a downtrend (bear market) according to an unobservable Markov chain. We consider a finite horizon investment problem and aim to maximize the expected return of the terminal wealth. We start by restricting to allowing flat and long positions only and describe the trading decisions using a sequence of stopping times indicating the time of entering and exiting long positions. Assuming trading all available funds, we show that the optimal trading strategy is a trend following system characterized by the conditional probability in the uptrend crossing two threshold curves. The thresholds can be obtained by solving the associated HJB equations. In addition, we examine trading strategies with short selling in terms of an approximation. Simulations and empirical experiments are conducted and reported.

**Keywords**: Trend following trading rule, bull-bear switching model, partial information, HJB equations

**AMS subject classifications**: 91G80, 93E11, 93E20

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# 1 Introduction

Roughly speaking, trading strategies can be classified as i) the buy and hold strategy, ii) the contra-trend strategy, and iii) the trend following strategy. The buy and hold strategy can be justified because the average return of stocks is higher than the bank rate[1]. An investor that performs the contra-trending strategy purchases shares when prices fall to some low level and sells when they go up to a certain high level (known as buy-low-sell-high). As the name suggests, the trend following strategy tries to enter the market in the uptrend and signal investors to exit when the trend reverses. In contrast to the contra-trend investors, a trend following believer often purchases shares when prices go up to a certain level and sells when they fall to a higher level (known as buy-high-sell-higher).

There is an extensive literature devoted to the contra-trend strategy. For instance, Merton [14] pioneered the continuous-time portfolio selection with utility maximization, which was subsequently extended to incorporate transaction costs by Magil and Constantinidies [13] (see also Davis and Norman [5], Shreve and Soner [19], Liu and Loeweinstein [12], Dai and Yi [3], and references therein). The resulting strategies turn out to be contra-trend because the investor is risk averse and the stock market is assumed to follow a geometric Brownian motion with constant drift and volatility. Recently Zhang and Zhang [24] showed that the optimal trading strategy in a mean reverting market is also contra-trend. Other work relevant to the contra-trend strategy includes Dai et al. [1], Song et al. [20], Zervors et al. [23], among others.

The present paper is concerned with a trend following trading rule. Traders who adopt this trading rule often use moving averages to determine the general direction of the market and to generate trade signals [21]. However, to the best of our knowledge, there is not yet any solid theoretical framework supporting the use of moving average[2]. Recently, Dai et al. [4] provided a theoretical justification of the trend following strategy in a bull-bear switching market and employed the conditional probability in the bull market to generate the trade signals. However, the work imposed a less realistic assumption[3]: Only one share of stock is allowed to be traded. In the present paper, we will remove this restriction and develop an optimal trend following rule. We also carry

[1] Recently Shiryaev et al. [18] provided a theoretical justification of the buy and hold strategy from another angle.

[2] There does exist research on statistical analysis for trading strategies with moving averages. See, for example, [6].

[3] The same assumption was imposed in [24], [20], and [23].

out extensive simulations and empirical analysis to examine the efficiency of our strategy.

Following [4], we model the trends in the markets using a geometric Brownian motion with regime switching and partial information. More precisely, two regimes are considered: the uptrend (bull market) and downtrend (bear market), and the switching process is modeled as a two-state Markov chain which is not directly observable[4]. We consider a finite horizon investment problem, and our target is to maximize the expected return of the terminal wealth. We begin by considering the case that only long and flat positions are allowed. We use a sequence of stopping times to indicate the time of entering and exiting long positions. Assuming trading all available funds, we show that the optimal trading strategy is a trend following system characterized by the conditional probability in the uptrend crossing two time-dependent threshold curves. The thresholds can be obtained through solving a system of HJB equations satisfied by two value functions that are associated with long and flat positions, respectively. Simulation and market tests are conducted to demonstrate the efficiency of our strategy.

The next logical question to ask is whether adding short will improve the return. Due to asymmetry between long and short as well as solvency constraint, the exact formulation with short selling still eludes us. Hence, we instead utilize the following approximation. First, we consider trading with the short and flat positions only. Using reverse exchange traded funds to approximate the short selling we are able to convert it to the case of long and flat. Then, assuming there are two traders A and B. Trader A trades long and flat only and trader B trades short and flat only. Combination of the actions of both A and B yields a trading strategy that involves long, short and flat positions. Simulation and market tests are provided to investigate the performance of the strategy.

The rest of the paper is arranged as follows. We present the problem formulation in the next section. Section 3 is devoted to a theoretical characterization of the resulting optimal trading strategy. We report our simulation results and market tests in Section 4. In Section 5, we examine the trading strategy when short selling is allowed. We conclude in Section 6. All proofs, some technical results and details on market tests are given in Appendix.

[4]Most existing literature in trading strategies assumes that the switching process is directly observable, e.g. Jang et al. [9] and Dai et al. [2].

## 2 Problem Formulation

Let $S_r$ denote the stock price at time $r$ satisfying the equation

$$dS_r = S_r[\mu(\alpha_r)dr + \sigma dB_r], \ S_t = X, \ t \le r \le T < \infty, \tag{1}$$

where $\alpha_r \in \{1, 2\}$ is a two-state Markov chain, $\mu(i) \equiv \mu_i$ is the expected return rate[5] in regime $i = 1, 2$, $\sigma > 0$ is the constant volatility, $B_r$ is a standard Brownian motion, and $t$ and $T$ are the initial and terminal times, respectively.

The process $\alpha_r$ represents the market mode at each time $r$: $\alpha_r = 1$ indicates a bull market (uptrend) and $\alpha_r = 2$ a bear market (downtrend). Naturally, we assume $\mu_1 > 0$ and $\mu_2 < 0$. Let $Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix}$, ($\lambda_1 > 0$, $\lambda_2 > 0$), denote the generator of $\alpha_r$. So, $\lambda_1$ ($\lambda_2$) stands for the switching intensity from bull to bear (from bear to bull). We assume that $\{\alpha_r\}$ and $\{B_r\}$ are independent.

Let

$$t \le \tau_1^0 \le v_1^0 \le \tau_2^0 \le v_2^0 \cdots \le \tau_n^0 \le v_n^0 \le \cdots, \ a.s.,$$

denote a sequence of stopping times. For each $n$, define

$$\tau_n = \min\{\tau_n^0, T\} \text{ and } v_n = \min\{v_n^0, T\}.$$

A buying decision is made at $\tau_n$ if $\tau_n < T$ and a selling decision is at $v_n$ if $v_n < T$, $n = 1, 2, \ldots$. In addition, we impose that one has to sell the entire shares by the terminal time $T$.

We first consider the case that the investor is either long or flat. If she is long, her entire wealth is invested in the stock account. If she is flat, all of her wealth is in the bank account that draws interests. Let $i = 0, 1$ denote the initial position. If initially the position is long (i.e, $i = 1$), the corresponding sequence of stopping times is denoted by $\Lambda_1 = (v_1, \tau_2, v_2, \tau_3, \ldots)$. Likewise, if initially the net position is flat ($i = 0$), then the corresponding sequence of stopping times is denoted by $\Lambda_0 = (\tau_1, v_1, \tau_2, v_2, \ldots)$.

Let $0 < K_b < 1$ denote the percentage of slippage (or commission) per transaction with a buy order and $0 < K_s < 1$ that with a sell order.

[5] Here we assume no dividend payments. If the stock pays a constant dividend yield, we then re-invest the dividends in the stock. So, our assumption is without loss of generality.

Let $\rho \geq 0$ denote the risk-free interest rate. Given the initial time $t$, initial stock price $S_t = S$, initial market trend $\alpha_t = \alpha \in \{1, 2\}$, and initial net position $i = 0, 1$, the reward functions of the decision sequences, $\Lambda_0$ and $\Lambda_1$, are the expected return rates of wealth:

$$
\begin{aligned}
&J_i(S, \alpha, t, \Lambda_i) \\
&= \begin{cases}
E_t\left\{ \log \left( e^{\rho(\tau_1 - t)} \prod_{n=1}^{\infty} e^{\rho(\tau_{n+1} - v_n)} \dfrac{S_{v_n}}{S_{\tau_n}} \left[ \dfrac{1 - K_s}{1 + K_b} \right]^{I_{\{\tau_n < T\}}} \right) \right\}, & \text{if } i = 0, \\
E_t\left\{ \log \left( \left[ \dfrac{S_{v_1}}{S} e^{\rho(\tau_2 - v_1)} (1 - K_s) \right] \prod_{n=2}^{\infty} e^{\rho(\tau_{n+1} - v_n)} \dfrac{S_{v_n}}{S_{\tau_n}} \left[ \dfrac{1 - K_s}{1 + K_b} \right]^{I_{\{\tau_n < T\}}} \right) \right\}, & \\
& \text{if } i = 1.
\end{cases}
\end{aligned}
$$

It is easy to see that

$$
\begin{aligned}
&J_i(S, \alpha, t, \Lambda_i) \\
&= \begin{cases}
E_t\left\{ \rho(\tau_1 - t) + \sum_{n=1}^{\infty} \left[ \log \dfrac{S_{v_n}}{S_{\tau_n}} + \rho(\tau_{n+1} - v_n) + \log \left( \dfrac{1 - K_s}{1 + K_b} \right) I_{\{\tau_n < T\}} \right] \right\}, & \text{if } i = 0, \\
E_t\left\{ \left[ \log \dfrac{S_{v_1}}{S} + \log(1 - K_s) + \rho(\tau_2 - v_1) \right] \right. & \\
\quad \left. + \sum_{n=2}^{\infty} \left[ \log \dfrac{S_{v_n}}{S_{\tau_n}} + \rho(\tau_{n+1} - v_n) + \log \left( \dfrac{1 - K_s}{1 + K_b} \right) I_{\{\tau_n < T\}} \right] \right\}, & \text{if } i = 1,
\end{cases}
\end{aligned}
$$

where the term $E \sum_{n=1}^{\infty} \xi_n$ for random variables $\xi_n$ is interpreted as $\limsup_{N \to \infty} E \sum_{n=1}^{N} \xi_n$. Our goal is to maximize the reward function[6].

**Remark 1** Note that the indicator function $I_{\{\tau_n < T\}}$ is used in the definition of the reward functions $J_i$. This is to ensure that if the last buy order is entered at $t = \tau_n < T$, then the position will be sold at $v_n \leq T$. The indicator function $I$ confines the effective part of the sum to a finite time horizon so that the reward functions are bounded above.

To exclude trivial cases,[7] we always assume

$$
\mu_2 - \frac{\sigma^2}{2} < \rho < \mu_1 - \frac{\sigma^2}{2}. \tag{2}
$$

Note that only the stock price $S_r$ is observable at time $r$ in marketplace. The market trend $\alpha_r$ is not directly observable. Thus, it is necessary to convert the problem into a completely observable one. One way to accomplish this is to use the Wonham filter [22].

[6]It is easy to see that the problem is equivalent to maximizing the expected logarithm utility of the terminal wealth over allowable trading strategies that incur proportional transaction costs.

[7]Intuitively, one should never buy stock if $\rho \geq \mu_1 - \frac{\sigma^2}{2}$ and never sell stock if $\rho \leq \mu_2 - \frac{\sigma^2}{2}$.

Let $p_r = P(\alpha_r = 1|\mathcal{S}_r)$ denote the conditional probability of $\alpha_r = 1$ (bull market) given the filtration $\mathcal{S}_r = \sigma\{S_u : \ 0 \le u \le r\}$. Then we can show (see Wonham [22]) that $p_r$ satisfies the following SDE

$$dp_r = [-(\lambda_1 + \lambda_2)\, p_r + \lambda_2]\, dr + \frac{(\mu_1 - \mu_2)p_r(1-p_r)}{\sigma} d\widehat{B}_r, \tag{3}$$

where $\widehat{B}_r$ is the innovation process (a standard Brownian motion; see e.g., Øksendal [15]) given by

$$d\widehat{B}_r = \frac{d\log(S_r) - [(\mu_1 - \mu_2)p_r + \mu_2 - \sigma^2/2]dr}{\sigma}. \tag{4}$$

It is easy to see that $S_r$ can be written in terms of $\widehat{B}_r$:

$$dS_r = S_r\,[(\mu_1 - \mu_2)\, p_r + \mu_2]\, dr + S_r \sigma d\widehat{B}_r. \tag{5}$$

Given $S_t = S$ and $p_t = p$, the problem is to choose $\Lambda_i$ to maximize the discounted return

$$J_i(S, p, t, \Lambda_i) \equiv J_i(S, \alpha, t, \Lambda_i),$$

subject to (3) and (5). We emphasize that this new problem is completely observable because the conditional probability $p_r$ can be obtained using the stock price up to time $r$.

Note that, for a given $\Lambda_0$, we have

$$\log \frac{S_{v_n}}{S_{\tau_n}} = \int_{\tau_n}^{v_n} f(p_r)dr + \int_{\tau_n}^{v_n} \sigma d\widehat{B}_r, \tag{6}$$

where

$$f(p_r) = (\mu_1 - \mu_2)p_r + \mu_2 - \frac{\sigma^2}{2}. \tag{7}$$

Note also that

$$E \int_{\tau_n}^{v_n} \sigma d\widehat{B}_r = 0. \tag{8}$$

Consequently, the reward function $J_0(S, p, t, \Lambda_0)$ can be rewritten completely as a function of $p_r$. Therefore, it is independent of the initial $S$, so is $J_1(S, p, t, \Lambda_0)$. In view of these, the corresponding value functions are functions of $(p, t)$.

For $i = 0, 1$, let $V_i(p, t)$ denote the value function with the states $p$ and net positions $i = 0, 1$ at time $t$. That is,

$$V_i(p, t) = \sup_{\Lambda_i} J_i(S, p, t, \Lambda_i).$$

The following lemma gives the bounds of the values functions. Its proof is given in Appendix.

**Lemma 1** *We have*

$$V_0(p,t) \geq \rho(T-t),\ \ V_1(p,t) \geq \log(1-K_s) + \rho(T-t),$$

*and*

$$V_i(p,t) \leq \left(\mu_1 - \frac{\sigma^2}{2}\right)(T-t), \quad \textit{for } i = 0, 1.$$

Next, we consider the associated Hamilton-Jacobi-Bellman equations. It is easy to see that, for $t < T$ and stopping times $\tau_1$ and $v_1$,

$$V_0(p,t) = \sup_{\tau_1} E_t \left\{ \rho(\tau_1 - t) - \log(1+K_b) + V_1(p_{\tau_1}, \tau_1) \right\}$$

and

$$V_1(p,t) = \sup_{v_1} E_t \left\{ \int_t^{v_1} f(p_s)ds + \log(1-K_s) + V_0(p_{v_1}, v_1) \right\},$$

where $f(\cdot)$ is as given in (7). Let $L$ denote the generator of $(t, p_t)$

$$L = \partial_t + \frac{1}{2}\left(\frac{(\mu_1-\mu_2)p(1-p)}{\sigma}\right)^2 \partial_{pp} + \left[-\left(\lambda_1+\lambda_2\right)p + \lambda_2\right]\partial_p.$$

Then, the associated HJB equations are

$$\begin{cases} \min\{-LV_0 - \rho, V_0 - V_1 + \log(1+K_b)\} = 0, \\ \min\{-LV_1 - f(p), V_1 - V_0 - \log(1-K_s)\} = 0, \end{cases} \tag{9}$$

with the terminal conditions

$$\begin{cases} V_0(p,T) = 0 \\ V_1(p,T) = \log(1-K_s). \end{cases} \tag{10}$$

Using the same technique as in Dai et al. [4], we can show that Problem (9)-(10) has a unique bounded strong solution $(V_0, V_1)$, where $V_i \in W_q^{2,1}([\varepsilon, 1-\varepsilon]\times[0,T])$, for any $\varepsilon \in (0, 1/2)$, $q \in [1, +\infty)$.

**Remark 2** In this paper, we restrict the state space of $p$ to $(0,1)$ because both $p = 0$ and $p = 1$ are entrance boundaries (see Karlin and Taylor [10] and Dai et al. [4] for definition and discussions).

Now we define the buying region $(BR)$, the selling region $(SR)$, and the no-trading region $(NT)$ as follows:

$$\begin{aligned} BR &= \left\{(p,t) \in (0,1)\times[0,T) : V_1(p,t) - V_0(p,t) = \log(1+K_b)\right\}, \\ SR &= \left\{(p,t) \in (0,1)\times[0,T) : V_1(p,t) - V_0(p,t) = \log\left(1-K_s\right)\right\}, \\ NT &= (0,1)\times[0,T)\diagdown\left(BR \cup SR\right). \end{aligned}$$

To study the optimal strategy, we only need to characterize these regions.

# 3 Optimal trading strategy

In this section, we present the main theoretical results.

**Theorem 2** *There exist two monotonically increasing boundaries $p_s^*(t)$, $p_b^*(t) \in C^\infty(0,T)$ such that*

$$\begin{aligned} SR &= \{p \in (0,1) \times [0,T) : p \le p_s^*(t)\}, \qquad (11)\\ BR &= \{p \in (0,1) \times [0,T) : p \ge p_b^*(t)\}. \qquad (12)\end{aligned}$$

*Moreover,*

*i)* $p_b^*(t) \ge \dfrac{\rho - \mu_2 + \sigma^2/2}{\mu_1 - \mu_2} \ge p_s^*(t)$ *for all* $t \in [0,T)$*;*

*ii)* $\lim_{t \to T^-} p_s^*(t) = \dfrac{\rho - \mu_2 + \sigma^2/2}{\mu_1 - \mu_2}$*;*

*iii) there is a* $\delta > \dfrac{1}{\mu_1 - \rho - \sigma^2/2} \log \dfrac{1+K_b}{1-K_s}$ *such that* $p_b^*(t) = 1$ *for* $t \in (T-\delta, T)$*.*

The proof is placed in Appendix. The theoretical results enable us to examine the validity of the program codes for numerically solving the system of HJB equations.

We call $p_s^*(t)$ ($p_b^*(t)$) the optimal sell (buy) boundary. To better understand Theorem 2, we provide a numerical result for illustration[8]. In Figure 1, we plot the optimal sell and buy boundaries against time. It can be seen that both the buy boundary and the sell boundary are increasing with time, between the two boundaries is the no trading region (NT), the buy region (BR) is above the buy boundary, and the sell region (SR) is below the sell boundary. Moreover, the sell boundary $p_s^*(t)$ approaches the theoretical value $\dfrac{\rho - \mu_2 + \sigma^2/2}{\mu_1 - \mu_2} = \dfrac{0.0679 + 0.77 + 0.184^2/2}{0.18 + 0.77} = 0.9$, as $t \to T = 1$. Also, we observe that there is a $\delta$ such that $p_b^*(t) = 1$ for $t \in [T-\delta, T]$, which indicates that it is never optimal to buy stock when $t$ is very close to $T$. Using Theorem 2, the lower bound of $\delta$ is estimated as

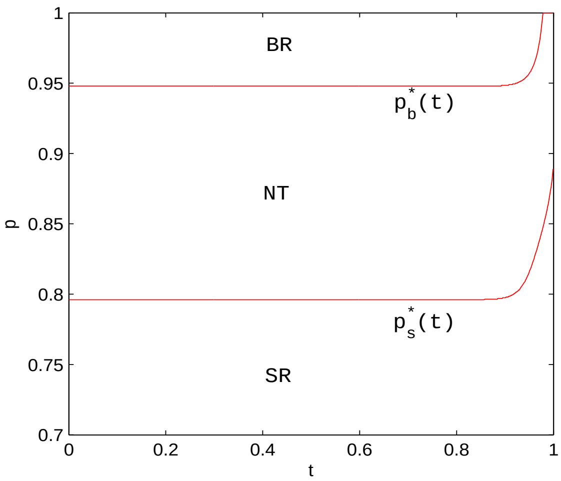
$$\frac{1}{\mu_1 - \rho - \sigma^2/2} \log \frac{1+K_b}{1-K_s} = \frac{1}{0.18 - 0.0657 - 0.184^2/2} \log \frac{1.001}{0.999} = 0.021,$$

which is consistent with the numerical result.

We now point out that our trading strategy is *trend following*. The numerical results illustrated in Figure 1 show that the thresholds $p_s^*(\cdot)$ and $p_b^*(\cdot)$ are rather close to constants except when $t$

[8] Analytic solutions of (9)-(10) are not available, but we can use numerical methods to find numerical solutions (cf. [4]).

Figure 1: Optimal trading strategy




Parameter values: $\lambda_1 = 0.36,\ \lambda_2 = 2.53,\ \mu_1 = 0.18,\ \mu_2 = -0.77,\ \sigma = 0.184,\ K_b = K_s = 0.001,\ \rho = 0.0679,\ T = 1.$

approaches $T$. The behavior of the thresholds when $t$ approaches to $T$ is due to our technical requirement of liquidating all the positions at $T$. Because we are interested in long-term investment, we will approximate these thresholds, as in [4], by constants $p_s^* = \lim_{T-t\to\infty} p_s^*(t)$ and $p_b^* = \lim_{T-t\to\infty} p_b^*(t)$.[9] Assume the initial position is flat and the initial conditional probability $p(0) \in (p_s^*, p_b^*)$. Then our trading strategy can be described as follows. As $p_t$ goes up to hit $p_b^*$, we take a long position, that is, investing all wealth in stock. We will not close out the position unless $p_t$ goes down to hit $p_s^*$. According to (3)-(4), we have

$$dp_r = g(p_r)dr + \frac{(\mu_1 - \mu_2)p_r(1-p_r)}{\sigma^2} d\log S_r, \tag{13}$$

where

$$g(p) = -\left(\lambda_1 + \lambda_2\right)p + \lambda_2 - \frac{(\mu_1 - \mu_2)p_t(1-p_t)\left((\mu_1 - \mu_2)p + \mu_2 - \sigma^2/2\right)}{\sigma^2}.$$

(13) implies that the conditional probability $p_t$ in the bull market increases (decreases) as the stock price goes up (down). Hence, our strategy suggests that we buy only when stock price is going up and sell only when stock price is going down. This is a typical trend following strategy!

We conclude this section by a verification theorem, showing that the solutions $V_0$ and $V_1$ of problem (9)-(10) are equal to the value functions and sequences of optimal stopping times can be

[9] The constant thresholds are essentially associated with the infinity horizon investment problem: $\lim_{T\to\infty} \frac{1}{T} \max E(W_T)$, where $W_T$ be the terminal wealth.

constructed by using $(p_s^*, p_b^*)$.

**Theorem 3** *(Verification Theorem) Let $(w_0(p,t), w_1(p,t))$ be the unique bounded strong solution to problem (9)-(10) with $w_0(p,t) \geq 0$ and $p_b^*(t)$ and $p_s^*(t)$ be the associated free boundaries. Then, $w_0(p,t)$ and $w_1(p,t)$ are equal to the value functions $V_0(p,t)$ and $V_1(p,t)$, respectively.*

*Moreover, let*

$$\Lambda_0^* = (\tau_1^*, v_1^*, \tau_2^*, v_2^*, \cdots),$$

*where the stopping times $\tau_1^* = T \wedge \inf\{r \geq t : \ p_r \geq p_b^*(r)\}$, $v_n^* = T \wedge \inf\{r \geq \tau_n^* : \ p_r \leq p_s^*(r)\}$, and $\tau_{n+1}^* = T \wedge \inf\{r > v_n^* : \ p_r \geq p_b^*(r)\}$ for $n \geq 1$, and let*

$$\Lambda_1^* = (v_1^*, \tau_2^*, v_2^*, \tau_3^*, \cdots),$$

*where the stopping times $v_1^* = T \wedge \inf\{r \geq t : \ p_r^* \leq p_s^*(r)\}$, $\tau_n^* = T \wedge \inf\{r > v_{n-1}^* : \ p_r \geq p_b^*(r)\}$, and $v_n^* = T \wedge \inf\{r \geq \tau_n^* : \ p_r \leq p_s^*(r)\}$ for $n \geq 2$. If $v_n^* \to T$, a.s., as $n \to \infty$, then $\Lambda_0^*$ and $\Lambda_1^*$ are optimal.*

The proof is in Appendix.

# 4 Simulation and market tests

We use both simulations and tests on historical market data to examine the effectiveness of the theoretical characterization of the trading strategy. To estimate $p_t$, the conditional probability in a bull market, we use a discrete version of the stochastic differential equation (13), for $t = 0, 1, \ldots, N$ with $dt = 1/252$,

$$p_{t+1} = \min\left(\max\left(p_t + g(p_t)dt + \frac{(\mu_1 - \mu_2)p_t(1-p_t)}{\sigma^2}\log(S_{t+1}/S_t), 0\right), 1\right), \tag{14}$$

where the price process $S_t$ is determined by the simulated paths or the historical market data. The min and max are added to ensure the discrete approximation $p_t$ of the conditional probability in the bull market stays in the interval $[0,1]$. Note that the equation for $p_t$ is the same as that in [4] because it is irrelevant to objective functions.

## 4.1 Simulations

For simulation we use the parameters summarized in Table 1 and a 40 year time horizon. They are the same as those used in [4] so that it is easy to compare the results.

| $\lambda_1$ | $\lambda_2$ | $\mu_1$ | $\mu_2$ | $\sigma$ | $K$ | $\rho$ |
|---|---|---|---|---|---|---|
| 0.36 | 2.53 | 0.18 | -0.77 | 0.184 | 0.001 | 0.0679 |

Table 1. Parameter values

Solving for the buy and sell thresholds using numerical solutions to the HJB equation, we derive $p_s^* = 0.796$ and $p_b^* = 0.948$. We run the 5000 round simulation for 10 times and summarize the mean of the total (annualized) return and the standard deviation in Table 2.

| | Trend Following | Buy and Hold | No. of Trades |
|---|---|---|---|
| Mean | 75.76(11.4%) | 5.62(4.4%) | 41.16 |
| Stdev | 2.48 | 0.39 | 0.29 |

Table 2. Statistics of ten 5000-path simulations

Comparing to the simulation results in [4] we only observe a slight improvement in terms of the ratio of mean return of the trend following strategy and the buy and hold strategy. However, the improvement is not significant enough to distinguish statistically from the results in [4] despite theoretically the present paper is more solid than [4]. This also reveals that using the conditional probability in the bull market as trade signals is rather robust against the change of thresholds.

The above simulation results are based on the average outcomes of large numbers of simulated paths. We now investigate the performance of our strategy with individual sample paths. Table 3 collects simulation results on 10 single paths using buy-sell thresholds $p_s^* = 0.795$ and $p_b^* = 0.948$ with the same data given in Table 1. We can see that the simulation is very sensitive to individual paths, but our strategy clearly outperforms the buy and hold strategy.

Note that this observation is consistent with the measurement of an effective investment strategy in marketplace. For example, O'Neil's CANSLIM works during a period of time does not mean it works on each stock when applied. How it works is measured based on the overall average when applied to a group of stocks fitting the prescribed selection criteria.

| Trend Following | Buy and Hold | No. of Trades |
|---|---|---|
| 67.080 | 3.2892 | 36.000 |
| 24.804 | 2.2498 | 42.000 |
| 22.509 | 0.40591 | 42.000 |
| 1887.8 | 257.75 | 33.000 |
| 26.059 | 0.16373 | 48.000 |
| 60.267 | 1.5325 | 43.000 |
| 34.832 | 5.7747 | 42.000 |
| 8.6456 | 0.077789 | 46.000 |
| 128.51 | 30.293 | 37.000 |
| 224.80 | 29.807 | 40.000 |

Table 3. Ten single-path simulations

## 4.2 Market tests

We now turn to test the trend following trading strategy in real markets. Here we conduct the ex-ante tests.[10] The parameters are determined using only information available at the decision time and updated periodically. More concretely, let us use the test of SP500 as an example to explain the process. We have SP500 historical closing data since 1962. We use the first 10 years data to derive statistics for bull (rally at least 20%) and bear (decline at least 20%) markets to serve as the initial parameters for determining the buy-sell thresholds. We then update the parameters and thresholds at the beginning of every year if new up or down trends are confirmed ending before the beginning of the year. We update the parameters using the so called exponential average method in which the update of the parameters is determined by the old parameters and new parameters with formula

$$update = (1 - 2/N)old + (2/N)new,$$

where we chose $N = 6$ based on the number of bull and bear markets between 1962–1972. The exponential average allows to overweight the new parameters while avoiding unwanted abrupt changes due to dropping old parameters. Then we use the yearly updated parameters to calculate the corresponding thresholds. Finally, we use these parameters and thresholds to test the SP500 index from 1972-2011. The equity curve of the trend following strategy is compared to the buy and hold strategy in the same period of time in Figure 2. The upper, middle and the lower curves represents the equity curves of the trend following strategy, the buy and hold strategy including dividend and the SP500 index without dividend adjustment, respectively.

As we can see, the trend following strategy not only outperforms the buy and hold strategy in total return, but also has a smoother equity curve, which means a higher Sharpe ratio. A similar ex-ante test is done for the Shanghai Stocks Exchange index (SSE). Since we have only 10 year data (2001–2011) for the SSE, we have to estimate the initial parameters. We summarize the tests on SP500 and SSE in Table 4 showing annualized return along with quarterly Sharpe ratio in parenthesis, where the estimate for the SSE initial parameters are $\mu_1 = 1$, $\mu_2 = -1$, and $\lambda_1 = \lambda_2 = 1$.[11] The equity curves for the SSE test are shown in Figure 3. The SSE index closing

[10] Ex-post tests of a trend following strategy were conducted in [4]. In the present paper we carry out the ex-ante tests which are more convincing.

[11] We have also tried other initial parameter values, e.g. $\mu_1 = 0.4$, $\mu_2 = -0.4$, $\lambda_1 = 1$, $\lambda_2 = 0.5$, which yield similar results as we present here. These results are available upon request.

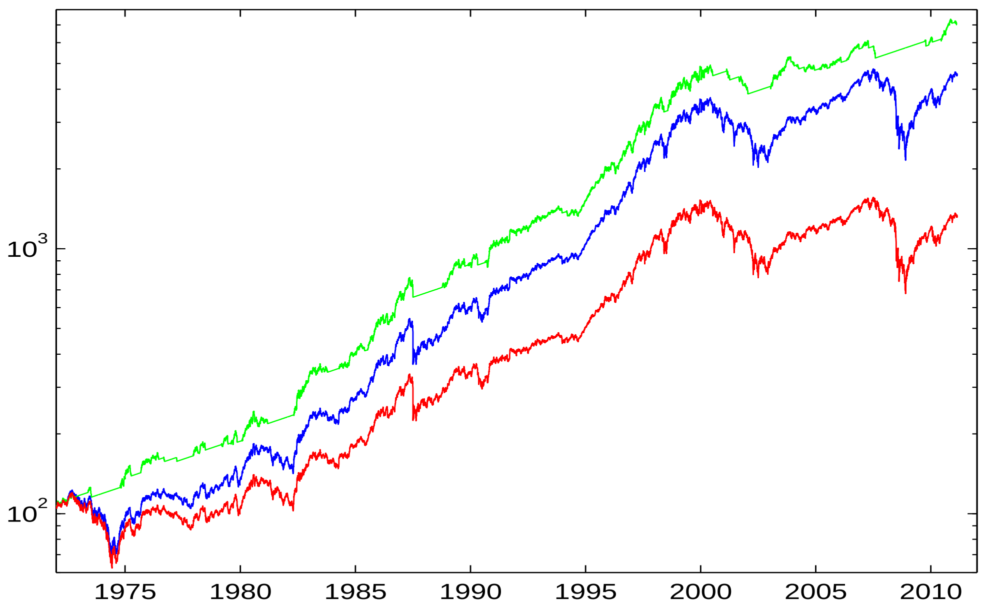



Figure 2: Trend following trading of SP500 1972–2011 compared with buy and hold

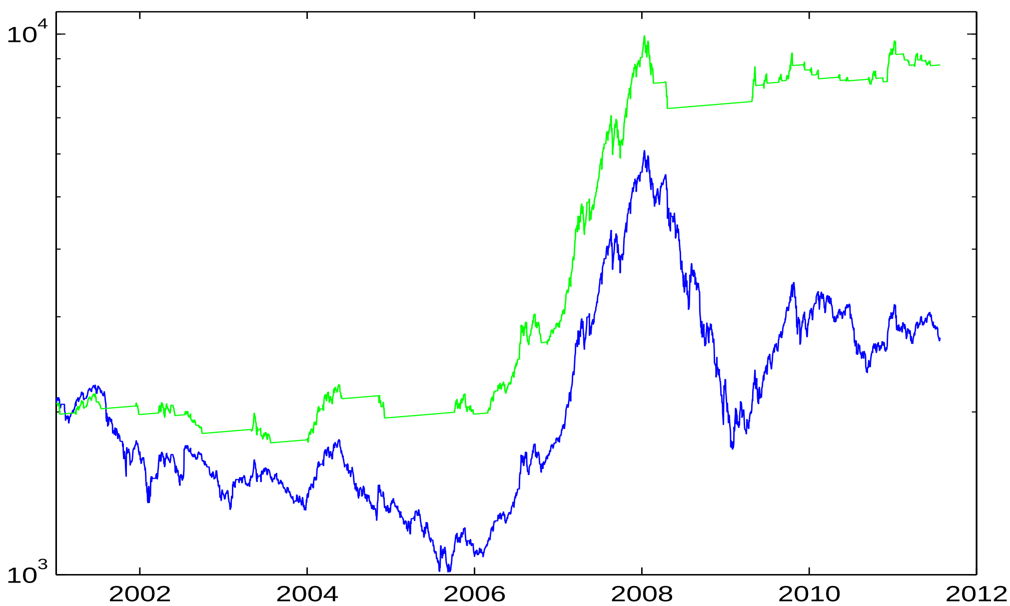



Figure 3: Trend following trading of SSE 2001–2011 compared with buy and hold

price has already adjusted for dividend. Thus, in Figure 3 we have only two curves: the upper represents the equity curve of the trend following method and the lower is that of the buy and hold strategy. The yearly parameters and thresholds for both SP500 and SSE tests are included in Tables in Appendix 6.2.

| Index(time frame) | Trend following | Buy and hold | 10 year bonds |
|---|---|---|---|
| SP500 (1972-2011) | 11.03%(0.217) | 9.8%(0.128) | 6.79% |
| SSE (2001-2011) | 14.0%(0.263) | 2.58%(0.083) | 3% |

Table 4. Testing results for trend following trading strategies

The test result for SP500 here is, if not better, at least comparable to the ex-pose test in [4] showing that trends indeed exists in the price movement of SP500.[12] The test on SSE shows that the trend also exists in China market.

As alluded to in the end of the last section, market trends are the consequence of aggregating many individual actions. Thus, it seems more stable than the generated paths using pure Brownian motions. Nevertheless, sensitivity with respect to parameters is still to be expected and putting test results into practice should be cautious.

# 5 Allowing shorts

Can we benefit by adding shorts? In practice there are differences between short and long. Notably, the short risks may lose more than the initial capital, so the no-bankruptcy constraint would have to be imposed, which makes the problem intractable under our theoretical framework. One way to circumvent this difficulty is to use the reverse Exchange Traded Funds (ETFs) of the corresponding indices to determine threshold values for short selling. In such an approximation to short sell $S_r$ we long a reverse ETF which is equivalent to longing an asset $\hat{S}_r = 1/S_r$. First we consider the case when only short and flat positions are allowed. Use $q_r = 1 - p_r$ to represent the conditional probability of $\hat{S}_r$ in an uptrend (equivalently $S_r$ is in a downtrend) and denote

$$\hat{\mu_1} = \sigma^2 - \mu_2, \hat{\mu_2} = \sigma^2 - \mu_1, \hat{\lambda_1} = \lambda_2, \hat{\lambda_2} = \lambda_1. \tag{15}$$

It is easy to verify that the process $\hat{S}_r$ and $q_r$ satisfy the system of stochastic differential equations

$$d\hat{S}_r = \hat{S}_r \left[ (\hat{\mu_1} - \hat{\mu_2})\, q_r + \hat{\mu_2} \right] dr + \hat{S}_r \sigma d\widehat{B}_r, \quad \hat{S}_t = \hat{S}, \tag{16}$$

[12] In [4], there is a mistake that the dividends are not treated as reinvestment. As a correction, the returns of the buy and hold strategy and the trend following strategy in [4] (Table 10) should be respectively 54.6 and 70.9, instead of 33.5 and 64.98, for SP500(1962-2008)

$$dq_r = \left[ -\left( \hat{\lambda_1} + \hat{\lambda_2} \right) q_r + \hat{\lambda_2} \right] dr + \frac{(\hat{\mu_1} - \hat{\mu_2}) q_r (1 - q_r)}{\sigma} d\widehat{B}_r, \quad q_t = q, \tag{17}$$

where

$$d\widehat{B}_r = -\frac{d \log(\hat{S}_r) - [(\hat{\mu_1} - \hat{\mu_2}) q_r + \hat{\mu_2} - \sigma^2/2] dr}{\sigma}. \tag{18}$$

Since the form of the system of stochastic equations (16) and (17) is the same as (3) and (5), we conclude that the optimal trading strategy when allowing only short and flat has the same form as that of the long and flat only case. Moreover, the thresholds for $q_r$ determined by (17) and (18) can be calculated using the same numerical procedure described in Section 4 with the parameter (15).

The next logical step is to allow both long and short along with flat positions in trading. Following the method we have used so far, to analyze trading with long, short and flat positions we would need to add a new value function $V_{-1}$ corresponding to start with a short position which will considerably complicate the analysis. An alternative is to consider the following approximation. First assume there are two traders A and B. Trader A trades long and flat only and trader B trades short and flat only. As discussed before, trader A can use the method in Section 3 to find two thresholds $p_b^*$ and $p_s^*$ and to make buying and selling decisions when $p_r$ cross those thresholds. Trader B can similarly determine two thresholds $q_b^*$ and $q_s^*$ for buying the inverse ETF and going flat when $q_r$ cross those thresholds. Furthermore, we can use the relationship $q = 1 - p$ to translate the thresholds for $q_r$ to that of $p_r$. Namely, when $p_r$ crosses $1 - q_b^*$ from above trader B should short the index and when $p_r$ crosses $1 - q_s^*$ from below trader B should go flat. Finally, we combine the action of A and B. Let us use the parameters in Table 1 as an illustration. In this case the four thresholds have the following order:

$$1 - q_b^* = 0.589 < p_s^* = 0.774 < 1 - q_s^* = 0.849 < p_b^* = 0.947. \tag{19}$$

Now we consider the net combined position of trader A and B. If we start with a long position one will sell to flat first when $p_r$ crosses $p_s^*$ from above. If market deteriorates further then one will sell short when $p_r$ crosses $1 - q_b^*$ eventually. Symmetrically, if the starting position is short one will cover the short first as market improves and $p_r$ crosses $1 - q_s^*$, and a long position will be initiated when the uptrend further strengthes such that $p_r$ crosses $p_b^*$.

We turn to simulations and tests with market data. We use actual short selling with 50% margin requirement. Thus, if the trend following strategy signals us to shot at $r = s$ and cover at $r = c$, we record a gain of

$$\frac{S_s + S_s(1-K) - S_c(1+K)}{S_s} \quad \text{(see Appendix 7.3)} \tag{20}$$

rather than simulate inverse ETF with a gain $S_s/S_c$. This is because the inverse ETF works well as an approximation of sell only in short term while our trend following trading strategy tends to have relative long holding period for a position.

Again, we run the 5000 round simulation for 10 times and summarize the mean and standard deviation in Table 5.

| | Trend Following | Buy and Hold | No. of Trades |
|---|---|---|---|
| Mean | 180.34 (13.9%) | 5.77 (4.5%) | 66.46 |
| Stdev | 1.61 | 0.168 | 0.124 |

Table 5. Statistics of ten 5000-path simulations allowing shorts

Comparing Table 5 and Table 2 we see that adding shorts does improve the performance considerably. However, it seems that the gain from short selling is less than that from long. This is not surprising given that the market is biased to the up side in the long run.

We now turn to market tests. Ex-ante test of SSE from 2001 to 2011 with long, short and flat positions yields an annualized return of 18.48% with a quarterly Sharpe ratio 0.306. This represents a significant improvement over the trend following strategy using only long and flat positions. However, a similar test for SP500 using the trend following strategy with long, short and flat positions from 1972 to 2011 only gives an annualized return of 2.57% which is worse than the annualized return of 8.57% using the buy and hold strategy. Thus, using the trend following strategy to sell index short may lose money. This is not entirely surprising. It is well known for practitioners that shorting a market is treacherous (see e.g. [16]).

# 6 Conclusion

We consider a finite horizon investment problem in a bull-bear switching market, where the drift of the stock price switches between two parameters corresponding to an uptrend (bull market) and a downtrend (bear market) according to an unobservable Markov chain. Our target is to maximize the expected return of the terminal wealth. We start by restricting to allowing flat and long positions only and use a sequence of stopping times to indicate the time of entering and exiting long positions. Assuming trading all available funds, we formulate the problem as a system of HJB equations satisfied by two value functions that are associated with long and flat positions, respectively. The system leads to two threshold curves that stand for the optimal buy and sell boundaries. We show that the optimal trading strategy is trend following and is characterized by the conditional probability in the uptrend crossing the buy and sell boundaries. We also examine trading strategies with short selling in terms of an approximation.

We carry out extensive simulations and empirical experiments to investigate the efficiency of our trading strategies. Here we have a number of interesting observations. First, simulations show that, somewhat surprisingly, the optimal trend following strategy only yields similar performance to the suboptimal trend following strategy derived in [4] where the conditional probability in bull market was also used to signal trading opportunities. This fact also demonstrates that using conditional probability in bull market as trade signals is robust and is insensitive to parameter perturbations. Second, the performances of the trend following strategy on individual simulated paths are dramatically different although averaging the results over large number of paths is rather stable. In almost all cases the trend following strategy apparently outperforms the buy and hold strategy. Third, the ex ante experiments with market data reveals that our strategy is efficient not only in U.S market (SP500 index) but also in China market (SSE index). Last but not the least, we observe an interesting divergence of the performances of the trend following trading strategy with short selling. Adding short selling significantly improves the performance in simulations but the performance in tests using the market historical data is mixed. In some cases it helps, and in some cases it actually hinders the performance. This is not entirely surprising because it is well known among practitioners that making money with short selling is difficult. This is an indication that up and downtrends in the real market are not symmetric and our regime switching model may only be a crude approximation to the real markets.

# 7 Appendix

## 7.1 Proofs of Results

**Proof of Lemma 1**. It is clear that the lower bounds for $V_i$ follow from their definition. It remains to show their upper bounds. Using (6) and (8) and noticing $0 \le p_r \le 1$, we have

$$\begin{aligned} E\left(\log \frac{S_{\upsilon_n}}{S_{\tau_n}}\right) &= E\left[\int_{\tau_n}^{\upsilon_n} f(p_r)dr\right] \\ &\le \left(\mu_1 - \frac{\sigma^2}{2}\right)\int_{\tau_n}^{\upsilon_n} dr = \left(\mu_1 - \frac{\sigma^2}{2}\right)(\upsilon_n - \tau_n). \end{aligned}$$

Note that $\log(1-K_s) < 0$ and $\log(1+K_b) > 0$. It follows that

$$\begin{aligned} J_0(S,p,t,\Lambda_0) &\le E_t\left\{\rho(\tau_1 - t) + \sum_{n=1}^{\infty}\left[\left(\mu_1 - \frac{\sigma^2}{2}\right)(\upsilon_n - \tau_n) + \rho(\tau_{n+1} - \upsilon_n)\right]\right\} \\ &\le \max\left\{\rho, \mu_1 - \frac{\sigma^2}{2}\right\}(T-t) \\ &= \left(\mu_1 - \frac{\sigma^2}{2}\right)(T-t), \end{aligned}$$

where the last equality is due to (2). We then obtain the desired result. Similarly, we can show the inequality for $V_1$. □

**Proof of Theorem 2**. Denote $Z(p,t) \equiv V_1(p,t) - V_0(p,t)$. It is not hard to verify $Z(p,t)$ is the unique strong solution to the following double obstacle problem:

$$\min\left\{\max\left\{-LZ - f(p) + \rho, Z - \log(1+K_b)\right\}, Z - \log(1-K_s)\right\} = 0,$$

in $(0,1) \times [0,T)$, with the terminal condition $Z(p,T) = \log(1-K_s)$. Apparently $\partial_t Z|_{t=T} \le 0$, which implies by the maximum principle

$$\partial_t Z \le 0. \tag{21}$$

Thanks to $f'(p) = \mu_1 - \mu_2 > 0$, we again apply the maximum principle to get

$$\partial_p Z \ge 0. \tag{22}$$

By (22), we immediately infer the existence of $p_s^*(t)$ and $p_b^*(t)$ as given in (9) and (10). Their monotonicity can be deduced from (21).

Now let us prove part i. If $(p,t) \in SR$, i.e. $Z(p,t) = \log(1-K)$, then

$$0 \le -L(\log(1-K)) - f(p) + \rho = -(\mu_1 - \mu_2)p - \mu_2 + \sigma^2/2 + \rho,$$

namely,

$$p \le \frac{\rho - \mu_2 + \sigma^2/2}{\mu_1 - \mu_2},$$

which is desired. The proofs of part ii and iii as well as the smoothness of $p_s^*(t)$ and $p_b^*(t)$ are similar to those in [4]. □

**Proof of Theorem 3**. First, note that $-\mathcal{L}w_0 - \rho \geq 0$. Using Dynkin's formula and Fatou's lemma as in Øksendal [15, p. 226], we have, for any stopping times $t \leq \theta_1 \leq \theta_2$, a.s.,

$$Ew_0(p_{\theta_1}, \theta_1) \geq E[\rho(\theta_2 - \theta_1) + w_0(p_{\theta_2}, \theta_2)]. \tag{23}$$

Similarly, using $-\mathcal{L}w_1 - f(p) \geq 0$, we have

$$Ew_1(p_{\theta_1}, \theta_1) \geq E\left[\int_{\theta_1}^{\theta_2} f(p_r)dr + w_1(p_{\theta_2}, \theta_2)\right] = E\left[\log \frac{S_{\theta_2}}{S_{\theta_1}} + w_1(p_{\theta_2}, \theta_2)\right]. \tag{24}$$

We next show, for any $\Lambda_1$ and $k = 1, 2, \ldots$,

$$\begin{aligned} Ew_0(p_{v_k}, v_k) \geq E\Bigg[&\rho(\tau_{k+1} - v_k) + \log \frac{S_{v_{k+1}}}{S_{\tau_{k+1}}} \\ &+ w_0(p_{v_{k+1}}, v_{k+1}) + (\log(1 - K_s) - \log(1 + K_b))I_{\{\tau_{k+1} < T\}}\Bigg]. \end{aligned} \tag{25}$$

In fact, using (23) and (24) and noticing that

$$w_0 \geq w_1 - \log(1 + K_b) \text{ and } w_1 \geq w_0 + \log(1 - K_s),$$

we have

$$\begin{aligned} &w_0(p_{v_k}, v_k) \\ \geq\ & E[\rho(\tau_{k+1} - v_k) + w_0(p_{\tau_{k+1}}, \tau_{k+1})] \\ \geq\ & E[\rho(\tau_{k+1} - v_k) + \left(w_1(p_{\tau_{k+1}}, \tau_{k+1}) - \log(1 + K_b)\right) I_{\{\tau_{k+1} < T\}}] \\ \geq\ & E[\rho(\tau_{k+1} - v_k) + \left(\log \frac{S_{v_{k+1}}}{S_{\tau_{k+1}}} + w_1(p_{v_{k+1}}, v_{k+1}) - \log(1 + K_b)\right) I_{\{\tau_{k+1} < T\}}] \\ \geq\ & E[\rho(\tau_{k+1} - v_k) + \left(\log \frac{S_{v_{k+1}}}{S_{\tau_{k+1}}} + w_0(p_{v_{k+1}}, v_{k+1}) + \log(1 - K_s) - \log(1 + K_b)\right) I_{\{\tau_{k+1} < T\}}] \\ =\ & E[\rho(\tau_{k+1} - v_k) + \log \frac{S_{v_{k+1}}}{S_{\tau_{k+1}}} + w_0(p_{v_{k+1}}, v_{k+1}) + (\log(1 - K_s) - \log(1 + K_b))\, I_{\{\tau_{k+1} < T\}}]. \end{aligned}$$

Note that the above inequalities also work when starting at $t$ in lieu of $v_k$, i.e.,

$$Ew_0(p_t, t) \geq E\left[\rho(\tau_1 - t) + \log \frac{S_{v_1}}{S_{\tau_1}} + w_0(p_{v_1}, v_1) + (\log(1 - K_s) - \log(1 + K_b))I_{\{\tau_1 < T\}}\right].$$

Use this inequality and iterate (25) with $k = 1, 2, \ldots$, and note $w_0 \geq 0$ to obtain

$$w_0(p, t) \geq V_0(p, t).$$

Similarly, we can show that

$$Ew_1(p_t, t) \geq E\left[\log \frac{S_{v_1}}{S_t} + w_1(p_{v_1}, v_1)\right] \geq E\left[\log \frac{S_{v_1}}{S_t} + w_0(p_{v_1}, v_1) + \log(1 - K_s)\right].$$

Use this and iterate (25) with $k = 1, 2, \ldots$ as above to obtain

$$w_1(p,t) \geq V_1(p,t).$$

Finally, it is easy to check that the equalities hold when $\tau_k = \tau_k^*$ and $v_k = v_k^*$. This completes the proof. $\Box$

### 7.2 Yearly parameters and thresholds in ex-ante tests

| date | $\mu_1$ | $\mu_2$ | $\lambda_1$ | $\lambda_2$ | $\sigma$ |
|---|---|---|---|---|---|
| 02-May-1972 | 0.2780 | -0.3400 | 0.8700 | 1.4600 | 0.1042 |
| 02-Jan-1973 | 0.2780 | -0.3400 | 0.8700 | 1.4600 | 0.0800 |
| 02-Jan-1974 | 0.2439 | -0.3400 | 1.0600 | 1.4600 | 0.1579 |
| 02-Jan-1975 | 0.2439 | -0.3480 | 1.0600 | 1.1660 | 0.2196 |
| 02-Jan-1976 | 0.2439 | -0.3480 | 1.0600 | 1.1660 | 0.1526 |
| 03-Jan-1977 | 0.2439 | -0.3480 | 1.0600 | 1.1660 | 0.1109 |
| 03-Jan-1978 | 0.2439 | -0.3480 | 1.0600 | 1.1660 | 0.0915 |
| 02-Jan-1979 | 0.2439 | -0.3480 | 1.0600 | 1.1660 | 0.1258 |
| 02-Jan-1980 | 0.2439 | -0.3480 | 1.0600 | 1.1660 | 0.1106 |
| 02-Jan-1981 | 0.2439 | -0.3480 | 1.0600 | 1.1660 | 0.1633 |
| 04-Jan-1982 | 0.2439 | -0.3480 | 1.0600 | 1.1660 | 0.1357 |
| 03-Jan-1983 | 0.2096 | -0.2905 | 0.7607 | 0.9727 | 0.1843 |
| 03-Jan-1984 | 0.2096 | -0.2905 | 0.7607 | 0.9727 | 0.1370 |
| 02-Jan-1985 | 0.2096 | -0.2905 | 0.7607 | 0.9727 | 0.1288 |
| 02-Jan-1986 | 0.2096 | -0.2905 | 0.7607 | 0.9727 | 0.1011 |
| 02-Jan-1987 | 0.2096 | -0.2905 | 0.7607 | 0.9727 | 0.1493 |
| 04-Jan-1988 | 0.2217 | -0.2905 | 0.5731 | 0.9727 | 0.3231 |
| 03-Jan-1989 | 0.2217 | -0.6240 | 0.5731 | 1.8316 | 0.1678 |
| 02-Jan-1990 | 0.2217 | -0.6240 | 0.5731 | 1.8316 | 0.1314 |
| 02-Jan-1991 | 0.2217 | -0.6240 | 0.5731 | 1.8316 | 0.1588 |
| 02-Jan-1992 | 0.2217 | -0.6240 | 0.5731 | 1.8316 | 0.1422 |
| 04-Jan-1993 | 0.2217 | -0.6240 | 0.5731 | 1.8316 | 0.0969 |
| 03-Jan-1994 | 0.2217 | -0.6240 | 0.5731 | 1.8316 | 0.0860 |
| 03-Jan-1995 | 0.2217 | -0.6240 | 0.5731 | 1.8316 | 0.0983 |
| 02-Jan-1996 | 0.2217 | -0.6240 | 0.5731 | 1.8316 | 0.0781 |
| 02-Jan-1997 | 0.2217 | -0.6240 | 0.5731 | 1.8316 | 0.1179 |
| 02-Jan-1998 | 0.2217 | -0.6240 | 0.5731 | 1.8316 | 0.1810 |
| 04-Jan-1999 | 0.2217 | -0.6240 | 0.5731 | 1.8316 | 0.2024 |
| 03-Jan-2000 | 0.2217 | -0.6240 | 0.5731 | 1.8316 | 0.1805 |
| 02-Jan-2001 | 0.2217 | -0.6240 | 0.5731 | 1.8316 | 0.2232 |
| 02-Jan-2002 | 0.2033 | -0.5113 | 0.4091 | 1.4462 | 0.2142 |
| 02-Jan-2003 | 0.3674 | -0.5692 | 1.4394 | 1.5773 | 0.2619 |
| 02-Jan-2004 | 0.3674 | -0.5692 | 1.4394 | 1.5773 | 0.1704 |
| 03-Jan-2005 | 0.3674 | -0.5692 | 1.4394 | 1.5773 | 0.1110 |
| 03-Jan-2006 | 0.3674 | -0.5692 | 1.4394 | 1.5773 | 0.1039 |
| 03-Jan-2007 | 0.3674 | -0.5692 | 1.4394 | 1.5773 | 0.1001 |
| 02-Jan-2008 | 0.3674 | -0.5692 | 1.4394 | 1.5773 | 0.1599 |
| 02-Jan-2009 | 0.2919 | -0.5748 | 1.0236 | 1.3483 | 0.4107 |
| 04-Jan-2010 | 0.8449 | -1.0038 | 3.4824 | 2.8989 | 0.2726 |
| 03-Jan-2011 | 0.8449 | -1.0038 | 3.4824 | 2.8989 | 0.1810 |

Table 6. Yearly parameters for the SP500 index ex-ante test

| Date | $p_s^*$ | $p_b^*$ |
|---|---|---|
| 02-May-1972 | 0.428 | 0.754 |
| 02-Jan-1973 | 0.391 | 0.774 |
| 02-Jan-1974 | 0.524 | 0.764 |
| 02-Jan-1975 | 0.580 | 0.770 |
| 02-Jan-1976 | 0.525 | 0.770 |
| 03-Jan-1977 | 0.481 | 0.783 |
| 03-Jan-1978 | 0.453 | 0.794 |
| 02-Jan-1979 | 0.498 | 0.777 |
| 02-Jan-1980 | 0.480 | 0.783 |
| 02-Jan-1981 | 0.535 | 0.769 |
| 04-Jan-1982 | 0.509 | 0.774 |
| 03-Jan-1983 | 0.566 | 0.768 |
| 03-Jan-1984 | 0.523 | 0.771 |
| 02-Jan-1985 | 0.514 | 0.773 |
| 02-Jan-1986 | 0.481 | 0.784 |
| 02-Jan-1987 | 0.535 | 0.769 |
| 04-Jan-1988 | 0.658 | 0.791 |
| 03-Jan-1989 | 0.656 | 0.873 |
| 02-Jan-1990 | 0.619 | 0.879 |
| 02-Jan-1991 | 0.648 | 0.874 |
| 02-Jan-1992 | 0.631 | 0.877 |
| 04-Jan-1993 | 0.570 | 0.890 |
| 03-Jan-1994 | 0.548 | 0.895 |
| 03-Jan-1995 | 0.572 | 0.890 |
| 02-Jan-1996 | 0.530 | 0.900 |
| 02-Jan-1997 | 0.602 | 0.883 |
| 02-Jan-1998 | 0.667 | 0.872 |
| 04-Jan-1999 | 0.683 | 0.872 |
| 03-Jan-2000 | 0.666 | 0.872 |
| 02-Jan-2001 | 0.698 | 0.873 |
| 02-Jan-2002 | 0.685 | 0.860 |
| 02-Jan-2003 | 0.572 | 0.769 |
| 02-Jan-2004 | 0.510 | 0.775 |
| 03-Jan-2005 | 0.449 | 0.797 |
| 03-Jan-2006 | 0.439 | 0.802 |
| 03-Jan-2007 | 0.433 | 0.804 |
| 02-Jan-2008 | 0.501 | 0.777 |
| 02-Jan-2009 | 0.739 | 0.856 |
| 04-Jan-2010 | 0.448 | 0.703 |
| 03-Jan-2011 | 0.395 | 0.725 |

Table 7. Yearly thresholds for the SP500 index ex-ante test

| date | $\mu_1$ | $\mu_2$ | $\lambda_1$ | $\lambda_2$ | $\sigma$ |
|---|---|---|---|---|---|
| 02-Jan-2001 | 1.0000 | -1.0000 | 1.0000 | 1.0000 | 0.1992 |
| 01-Jan-2002 | 0.7259 | -1.0000 | 1.3846 | 1.0000 | 0.2126 |
| 01-Jan-2003 | 0.6694 | -0.9199 | 1.6290 | 1.1950 | 0.2333 |
| 01-Jan-2004 | 0.6694 | -0.7869 | 1.6290 | 1.4478 | 0.1707 |
| 03-Jan-2005 | 0.5279 | -0.7869 | 1.3429 | 1.4478 | 0.1977 |
| 02-Jan-2006 | 0.5279 | -0.6607 | 1.3429 | 1.2205 | 0.2124 |
| 01-Jan-2007 | 0.5279 | -0.6607 | 1.3429 | 1.2205 | 0.2086 |
| 01-Jan-2008 | 0.6185 | -0.6607 | 1.0374 | 1.2205 | 0.3399 |
| 05-Jan-2009 | 4.8903 | -1.8062 | 23.4833 | 4.8778 | 0.4485 |
| 04-Jan-2010 | 3.6021 | -2.3256 | 16.1120 | 7.6729 | 0.3017 |
| 04-Jan-2011 | 2.7612 | -1.7366 | 12.2969 | 5.6829 | 0.2270 |

Table 8. Yearly parameters for the SSE index ex-ante test

| Date | $p_s^*$ | $p_b^*$ |
|---|---|---|
| 02-Jan-2001 | 0.361 | 0.682 |
| 01-Jan-2002 | 0.455 | 0.742 |
| 01-Jan-2003 | 0.474 | 0.738 |
| 01-Jan-2004 | 0.404 | 0.721 |
| 03-Jan-2005 | 0.487 | 0.759 |
| 02-Jan-2006 | 0.464 | 0.721 |
| 01-Jan-2007 | 0.462 | 0.722 |
| 01-Jan-2008 | 0.483 | 0.679 |
| 05-Jan-2009 | 0.186 | 0.442 |
| 04-Jan-2010 | 0.251 | 0.592 |
| 04-Jan-2011 | 0.233 | 0.604 |

Table 9. Yearly thresholds for the SSE index ex-ante test

### 7.3 Return on short selling

Assuming that the signals from the trend following strategy indicate short selling at $r = s$ and then cover at $r = c$ and that the margin requirement for short selling is $\alpha$, we calculate the return on the above trade when trading full margin. Let $w$ be the wealth at $r = s$. Suppose we can short sell $k$ shares on full margin. Then $kS_s = \alpha(w + kS_s)$ or $w = kS_s\frac{1-\alpha}{\alpha}$. When we cover at $r = c$ the net gain is $kS_s(1 - K) - kS_c(1 + K)$ taking into account of the trading cost. Thus, the return is

$$\frac{w + kS_s(1 - K) - kS_c(1 + K)}{w} = \frac{S_s(1 - \alpha) + S_s(1 - K)\alpha - S_c(1 + K)\alpha}{S_s(1 - \alpha)}.$$

When $\alpha = 50\% = 1/2$ we get (20).

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